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FOUR-KING CHESS WITH DICE IS NEITHER UNREALISTIC NOR MESSED UP

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Abstract

Kautilya's maṇḍala model has intrigued indologists and political scientists for some time. It deals with friendship and enmity between countries that are direct or indirect neighbours. (Ghosh; 1936) suggests a close relationship between this model and Indian four-king chess. We try to corroborate his claim by presenting a stylized game-theory model of both Indian four-king chess and *Kautilya's maṇḍala* theory. Within that game model, we can deal with *Kautilya's* conjecture according to which an enemy's enemy is likely to be one's friend. Arguably, this conjecture is reflected in the ally structure of four-king chess. We also comment on the widespread disapproval of dice in (four-king) chess.

1 Introduction

Chess seems to have originated in India, perhaps in the fifth century C.E.¹ Chess boards used to have 8 lines and 8 columns early on and each of the two players commanded 16 pieces. The Sanskrit term for that game is *caturāṅga* which means “[boardgame] with four parts”. The four parts refer to four different kind of troops in real life or pieces used in Indian chess, for example elephants, chariots, horses, and infantry (see (Scharfe; 1989, pp. 186-199) and (Bock-Raming; 1996, p. 1), respectively). Apart from these four pieces, there was a king. Let us call this sort of chess (with many different manners of

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¹Chess history, in particular in relation to India, is discussed by van der Linde (1874, 1881), (Murray; 1913, chapters II-IV), (Petzold; 1986, pp. 17-40), Thieme (1962, 1994), Syed (1993, 1995, 2001), (Parlett; 1999), Bock-Raming (1996, 2001), (Schädler; 1999), and (Schlingloff; 2009) while related dice games are treated by (Lüders; 1907).

play, to be sure) “two-king chess”. A variant with four armies each consisting of 8 pieces was also in use. It is also called *caturāṅga* or, tellingly, *catūrāṅjī* (“[boardgame] with four kings”). This four-king chess seems to have been played by four players, but perhaps also by two players each commanding two armies. In some versions of chess, dice are used to tell the players which pieces to move.

Most later chess historians surmise that four-king chess has developed out of two-king chess. The priority question² is not relevant for this paper. Instead, we are concerned with a somewhat odd aspect of this debate. Four-king chess with dice is perceived negatively and has been called “unrealistic”³ or “messed up”⁴. The criticism is directed at (I) the use of dice within (at least some versions of) this kind of chess and (II) the fact of using four armies fighting against each other. With respect to (I), (van der Linde; 1874, p. 80) had expressed his dismay at chess with dice by claiming: “Throwing dice and engaging in logical reasoning are absolutely heterogenous so that an invention with such an irreconcilable contradiction should be considered a psychological impossibility.”⁵ Turning to (II), we follow Maria Schetelich (personal communication) and take exception to (Thieme; 1994, p. 19) who, supporting a personal communication by Renate Söhnen, finds the following account probable: Chariots stopped to play an important role in Indian warfare and the chess piece “chariot” was eventually replaced by the chess piece “boat”. With the old formation not in place any more, *caturāṅga* (“with four parts”) was misunderstood in the sense of “four parties” and arising from that misunderstanding four-king chess was constructed. According to this argument, a new kind of game (four-king chess) was conceived because the

²While (Ghosh; 1936, pp. xviii-xix), (Petzold; 1986, pp. 20-40), and others argue for the priority of four-king chess, other chess historians, such as (Murray; 1913, p. 75), (Thieme; 1994, pp. 18-19), or (Syed; 1995, pp. 67-70) consider two-king chess the older variant.

³(Thieme; 1994, p. 19) opines that four-king chess is an unrealistic construction (“von [...] wirklichkeitsfremder Konstruktion”). He is seconded by (Syed; 1995, p. 68).

⁴(van der Linde; 1881, p. 259) uses the German word “verpfuscht” and cites the Medieval pseudo-Ovidian poem “de Vetula” (edited by (Robathan; 1968)) where, in the first book, we have “*cum deciis [...], qui primus lusit in illo, fedavid ludum*” (Robathan; 1968, v. 654-655, p. 72). The *ludus* is called *ludus scacorum* (Robathan; 1968, v. 600, p. 71) before. Thus, we can translate as “the first who has played with dice in chess has disfigured it”. (Murray; 1913, p. 508) praises the poem’s author’s (presumably some Richard de Fournival) “condemnation of the use of dice” as being “in advance of his time”. The negative attitude towards dice games was also typical of the Medieval Catholic Church, as is clear from (Murray; 1913, pp. 410–411, fn. 49, 50).

⁵Translated from the German original: “Würfeln und Combiniren sind hier so absolut heterogen, dass eine mit einem solchen unversöhnlichen Widerspruch behaftete Erfindung, für eine psychologische Unmöglichkeit gehalten werden muss.”

description of another game (two-king chess) was not as “realistic” as before. We find this unconvincing.

Most chess historians agree that chess is a war game, used for didactic purposes (see, for example, (Syed; 1995, p. 67) and also the conclusion). If so, why should we not consider four-king chess a reflection of a simple *maṇḍala* model with four parties? In his effort to date four-king chess, (Ghosh; 1936, p. xxv) makes this connection.

The *maṇḍala* theory is known to us through the *Arthaśāstra*, which was probably written by *Kauṭilya* (roughly 2000 years ago, consult (Olivelle; 2013, p. 29)).⁶ The Sanskrit word *maṇḍala* means “circle, wheel”. In *Kauṭilya*’s *Arthaśāstra* it refers to a ringlike structure of countries. A king should envision his country at the center. This king is then called *vijigīṣu* or “seeker after conquest”: “The whole point of being a king was to expand his territory and treasury by conquest. But, of course, all the neighboring kings were operating under the same assumption” (see (Olivelle; 2013, p. 47)). *Kauṭilya*’s main theoretical idea was simple and intriguing. War can only be waged with (direct) neighbors (local warfare).⁷ Therefore, neighbors tend to be enemies. Also, since these enemies might be attacked from the other side, the enemies of enemies tend to be friends. While this *Kauṭilyan* conjecture surely has a lot of intuitive appeal, we are not aware of any formal model that confirms or disproves it.⁸ Providing the building blocks for analyzing *Kauṭilya*’s conjecture is a central aim of this paper. We will see how *Kauṭilya*’s conjecture relates to the question of how kings are allied in four-king chess.

Due to lack of historical evidence, the specific nature of the link between the *maṇḍala* model and four-king chess cannot be ascertained. Thus we cannot argue for this strong claim: “The *maṇḍala* model was developed first. In order to understand its working, four-king chess was invented.” We rather

⁶A later exposition of *maṇḍala* theory is found in the *Nītisāra* (see (Mitra; 1982)), a political treatise that is several hundred years younger than the *Arthaśāstra*. A recent paper on this important text is (Singh; 2010). (Schetelich; 1997) explores the changes undergone by the concept of *maṇḍala* within the *Arthaśāstra* and also in *Dharmaśāstras* and other texts during the centuries.

⁷We do not claim that *Kauṭilya* disregards wars between non-contiguous kingdoms. After all, there may well have been stripes of land not belonging to any kingdom so that an attack does not, of necessity, have to be waged from one’s own territory (or the territory of a friend). Notwithstanding this caveat, we think that local warfare is a quite natural assumption (see also (Olivelle; 2013, p. 48)).

⁸An empirical analysis is presented by (Maoz et al.; 2007) who do not, by the way, refer to the *Arthaśāstra*. Roughly speaking, the empirical findings are in support of the conjecture, but the authors also obtain some counterintuitive results.

argue for the following weak version: “The *maṇḍala* model and four-king chess show striking similarities. Four-king chess was used in order to teach Indian kings and princes some of the strategic knowledge inherent in the *maṇḍala* model.”⁹

Thus, the aim of our paper is to defend four-king chess against Thieme, van der Linde, and others. Our paper is organized as follows. In the next section, we briefly mention the evidence of how the four armies were distributed on the chess board and on the possibility that two of these four kings were allied, fighting against the other two kings. Section 3 then presents a few quotes from *Kauṭilya’s Arthaśāstra*. Section 4 develops a formal model that allows to express and analyze *Kauṭilya’s* conjecture. Finally, section 5 concludes.

2 Four-king chess with dice

We have some evidence on how four-king chess was played. We cite from the *Tithitattva* (probably 16th century CE, edited and translated into German by (Weber; 1873)). All the verses cited are also found (with minor differences) in the *Caturaṅgadīpikā* (edited and translated into English by (Ghosh; 1936)¹⁰). The *Mānasollāsa* (12th century CE)¹¹ also has a few verses on four-king chess. Of course, quite a number of lacunae remain. We now focus on those aspects important for the present paper.

2.1 Distributing the four armies on the chess board

The *Tithitattva*¹² explains how the four armies with their four kings are distinguished by color and how they are located on a chess board (see Figure 1):

⁹(Ghosh; 1936, p. xxv) observes that more complicated models (with more than four countries) are described in the *Arthaśāstra* and elsewhere. The author then goes on to conclude that four-king chess must be older than the *Arthaśāstra* because more complicated *maṇḍalas* can be assumed to have developed from more simple ones. However, chess as a didactic version of warfare in a *maṇḍala*, may well be modeled on simple rather than complicated *maṇḍalas*. Also, if one finds the development from simple to more complicated structures plausible, one has an additional argument against four-king chess developing into two-king chess (against Ghosh’s own conviction).

¹⁰The numbering differs because the *Caturaṅgadīpikā* cites from, and comments upon, the *Tithitattva* and other sources.

¹¹It has been edited by Shrigondekar in 1925, 1939 and 1961. In the third volume, chess is dealt with in 5.560-623 and four-king chess in 5.615-623. The whole chess portion has been translated into German twice and independently by (Syed; 1993) and (Bock-Raming; 1996).

¹²(Weber; 1873, |2|-|3|, p. 64)

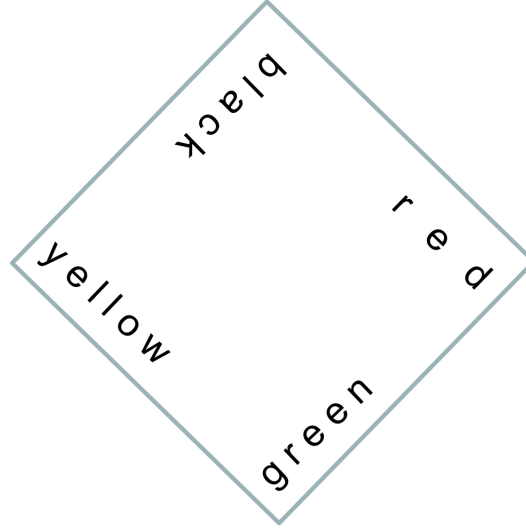


Figure 1: Four-player chess

aṣṭau koṣṭhān samālikhya pradakṣiṇakrameṇa tu |
aruṇam pūrvataḥ kṛtvā dakṣiṇe haritam balam ||2||
Pārtha paścimataḥ pītam uttare śyāmalam balam |
... ||3||

OR

“Having drawn the eight fields, but having placed in clockwise fashion¹³ the red in front, the green army in the south, and, Pārtha, the yellow one in the west and the black army in the north.”

¹³Weber’s German translation pulls *pradakṣiṇakrameṇa* into the absolute construction (“Acht Felder zeichne man der Reihe nach nach rechts hin”) while Murray ignores *pradakṣiṇakrameṇa* and (Ghosh; 1936, v. 8) translates as “Drawing eight squares (on each side), place the pieces (*balas*) from the right. Put the red pieces in the east ...”. Note also (Thieme; 1962, pp. 204-208) who also translates as “clockwise”, but with respect to the movement of pieces.

2.2 Using the dice to determine the pieces

After explaining how the pieces are ordered within each army¹⁴, the *Tithi-tattva*¹⁵ explains the use of dice:

... /
pañcakena vaṭī rājā, catuṣkeṇaiva kuñjaraḥ ||5|||
trikeṇa tu calaty aśvaḥ, Pārtha, naukā dvayena tu /
 ... ||6|||
 or

“With a five [on the dice] a pawn and a king move, with just a four the elephant, but with a three the horse, Pārtha, the boat, again, with a two.”

(Lüders; 1907, p. 69) thinks that a *pāśaka* was used whose four sides were indicated by the numbers 5 through 2.

2.3 Alliances

It may well have been the case that the pawns (that march forward, similar to modern chess) move towards the next clockwise army (compare Figure 1). Indeed, chess historians such as (Murray; 1913, p. 69) depict the pieces on the chess board in this manner. Then, the red army should tend to attack the green one, the green one the yellow one and so forth in a clockwise fashion. Note, however, that we are not aware of any scriptural or pictorial evidence to that effect.

It may well be that Murray’s (1913, p. 72) claim (regarding the *Tithi-tattva*) is related: “A game is played by four players allied in pairs. In the poem, red and yellow are allies, green and black. The nature of the alliance does not clearly transpire: it can hardly have been very cordial and sincere, when it was equally profitable to capture the ally’s King or an enemy’s King, and a necessity for the gain of the most profitable victory.” Apart from the pawns’ movements, the pairing claimed by Murray (where a king’s ally is found across the diagonal) makes perfect sense from the point of view of this paper.

¹⁴Starting from the corners, one has the boat, the horse, the elephant, and the king along the respective first line and four pawns in front on the second line. This is described in (Weber; 1873, |3|-|5|, pp. 64-65). In real warfare, “the king did not usually fight in the frontline” according to (Scharfe; 1989, p. 181).

¹⁵(Weber; 1873, |5|-|6|, p. 65)

This is how the ally is mentioned in the *Tithitattva*¹⁶:

*mitrasinhāsanam Pārtha yadārohati*¹⁷ *bhūpatiḥ* |
tadā sinhāsanam nāma sarvaṃ nayati tadbalam ||16||

or

“When a king, Pārtha, ascends the throne of an ally, then the so-called [procedure¹⁸] *siṃhāsana* [takes effect] and he leads his [the ally’s] whole army.”

Alliances are even more explicit in the *Mānasollāsa* 5.615-6:

... |
*catvāro khelakhā*¹⁹ *yatra tatra vyūho nirūpyate* ||615||
ekāntaram prakartavyaṃ pāṇḍuram lohitaṃ balam.. |
 ... ||616||

or

“Where the four players [sit], the troops are positioned. The white and the red army are to be placed alternatively²⁰.”

Thus, the ally structure is clearly stated in the *Mānasollāsa*.

3 *Kauṭilya* on war and peace

3.1 Dice representing fortune of war

Taking up the two criticisms on four-king chess with dice, we need to refer to (I) the use of dice within (at least some versions of) this kind of chess and (II) the fact of using four armies fighting against each other.

¹⁶(Weber; 1873, [16], p. 72)

¹⁷Weber has *yadā ’rohati*. It seems inconsequential whether we understand *rohati* or *ārohati*.

¹⁸In (Weber; 1873, [9], p. 69), *siṃhāsana* is listed among the seven *pracāraḥ* while (Ghosh; 1936, 15., pp. 3, 8) has *prakāraḥ*, instead. Both terms might be translated as “procedures” and mostly refer to “manners of winning”.

¹⁹This is a sensible emendation for *lekhakhā*.

²⁰*ekāntara* means the “other than the one [just placed]”. (Syed; 1993, p. 102) comments that the two armies belonging to one colour are to be placed diagonally. Tellingly, the same word is used by KAS 6.2.15 (see subsection 3.2 below).

We argue that dice represent the uncertainty inherent in matters of war and peace. *Kauṭilya* was well aware of this uncertainty. Let us consider two passages from the war-and-peace part of the *Arthaśāstra* (which we refer to by KAŚ). In KAŚ 6.2.6-12 ((Kangle; 1969, p. 165) and (Olivelle; 2013, p. 273)), we have:

mānuṣaṃ nayāpanayau, daivam ayānayau |6|
daivamānuṣaṃ hi karma lokam yāpayati |7|
adr̥ṣṭakāritaṃ daivam |8|
tasminn iṣṭena phalena yogo 'yah, aniṣṭenānayaḥ |9|
dr̥ṣṭakāritaṃ mānuṣaṃ |10|
tasmin yogakṣemaniṣpattir nayaḥ, vipattir apanayaḥ |11|
tac cintyam, acintyam daivam |12|

or

“Good and bad policy pertain to the human realm, while good and bad fortune pertain to the divine realm. Divine and human activity, indeed, makes the world run. The divine consists of what is caused by an invisible agent. Of this, attaining a desirable result is good fortune, while attaining an undesirable result is bad fortune. The human consists of what is caused by a visible agent. Of this, the success of enterprise and security is good policy, while their failure is bad policy. This is within the range of thought, whereas the divine is beyond the range of thought.”

One may argue that this quote is not a good example of the awareness of uncertainty. After all, the divine is largely incidental to the treatise. While this is true, the quote clearly brings out the fact that humans have partial control only over matters of war and peace (and others also). The second passage is from KAŚ 9.7.8-10 ((Kangle; 1969, p. 289) and (Olivelle; 2013, p. 367)):

tayoḥ artho na veti, anartho na veti, artho 'nartha iti, anartho 'rtha iti
saṃśayaḥ |8|
śatrumitram utsāhayitum artho na veti saṃśayaḥ |9|
śatrubalam arthamānābhyaṃ āvāhayitum anartho na veti saṃśayaḥ |10|

or

“Between these two, when one questions: ‘Is this an advantage or not?’ ‘Is this a disadvantage or not?’ ‘Is this advantage actually a

disadvantage?’ ‘Is this disadvantage actually an advantage?’—it is uncertainty. There is uncertainty as to whether it is an advantage or not to rouse up an ally of the foe. There is uncertainty as to whether it is a disadvantage or not to entice troops of the foe with money and honors.”

3.2 Modelling *Kauṭilya’s maṇḍala*

Turning to (II), one needs to observe that *Kauṭilya’s maṇḍala* theory consists of two parts, the *maṇḍala* and the policies to be pursued within this circle.²¹ First, in book 6 (immediately following the first quote from the previous section), *Kauṭilya* describes the *maṇḍalas* which allow to identify enemies and friends in a straightforward manner. KAŚ 6.2.13-15 ((Kangle; 1969, p. 165) and (Olivelle; 2013, p. 274)) has

rājā ātmadravyaprakṛtisampanno nayasyādhiṣṭhānaṃ vijigīṣuḥ /13/
tasya samantato maṇḍalābhūtā bhūmyanantarā aripakṛtiḥ /14/
*tathaiiva bhūmyekāntarā*²² *mitrapakṛtiḥ* /15/

or

“The seeker after conquest is a king who is endowed with the exemplary qualities both of the self and of material constituents, and who is the abode of good policy. Forming a circle all around him and with immediately contiguous territories is the constituent comprising his enemies. In like manner, with territories once removed from his, is the constituent comprising his allies.”

For our paper, this is the central quotation. Its relevance is twofold:

- We have here a definition of a *maṇḍala*.²³ In the next section, we formally define *maṇḍalas*.
- These quotations are often summarized by “the enemy of my enemy is my friend”. This is what we call *Kauṭilya’s* conjecture.

²¹(Scharfe; 1989, pp. 202-212) explains both parts within the state’s foreign affairs.

²²*ekāntara* is also found in subsection 2.3 above.

²³(Scharfe; 1968, p. 126) presents a linear depiction of a king’s *maṇḍala*. He argues that such a scheme is geared towards individual conflicts. This discussion is not relevant for the current paper.

Second (from book 7 onward), *Kauṭilya* expounds six *guṇas* (“strategies”): *saṃdhi* (“peace pact”), *vigraha* (“initiating hostilities”)²⁴, *āsana* (“remaining stationary”), *yāna* (“marching into battle”), *saṃśraya* (“seeking refuge”), and *dvaidhībhāva* (“double stratagem”).²⁵ (Olivelle; 2013, p. 659) explains the translation of *guṇa* as strategy by referring to KAŚ 6.2.6-12 from the previous section: While the literal meaning of *guṇa*²⁶ is “quality or attribute”, we are concerned here with the six attributes of (good) human policy. This is an important remark for us: A king may follow the recommendations given by *Kauṭilya*, but he should still be aware of the “divine” and “invisible” (the dice!) that may also feed into success or failure.

Taking up two (arguable central) strategies, we provide a game-theoretic analysis of the *maṇḍala* model. We focus on “remaining stationary” and “marching into battle” and translate these actions with “not attacking” and “attacking”, respectively. We are then in a position to define a game in the sense of game theory. Within this game, we can find out whether the enemy of an enemy is, indeed, a friend.

4 Fighting involving friends and enemies

4.1 Neighborhood structures and fighting structures

We now provide a formal definition of a *maṇḍala* by help of the concept of a “neighborhood structure”. Although we later focus on four countries, the following definitions allow to define a “neighborhood structure” for any number of countries. We express the fact that countries i and j are neighbors by way of the shorthand $i - j$ or $j - i$. Thus, $i - j = j - i$.

Definition 1. Let $I = \{1, \dots, n\}$ be a set of n countries. A neighborhood structure N on I is a subset of $I^{(2)} := \{i - j : i, j \in I, i \neq j\}$.

- By $N(i) = \{i - j : j \in I, i - j \in N\}$ we understand the set of links that country i entertains under the neighborhood structure N .

²⁴(Olivelle; 2011) convincingly defends his understanding of *vigraha* (“initiating hostilities”) versus *yāna* (“marching into battle”). He writes: “... a state of *vigraha* may not result in actual fighting, but rather weaken the enemy by one’s ability to resist his attacks and by destroying his sources of income, much like today’s special operations” (p. 135).

²⁵See KAŚ 7.1.2 ((Kangle; 1969, p. 168) and (Olivelle; 2013, p. 277))

²⁶Literally, KAŚ 7.1.1 (introducing the list) reads *śaḍguṇasya prakṛtimaṇḍalaṃ yoniḥ* ((Kangle; 1969, p. 168)), translated as “The basis of the sixfold strategy is the circle of constituents” by (Olivelle; 2013, p. 277).

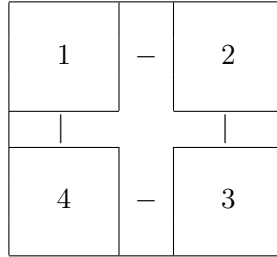


Figure 2: A symmetric four-country neighbourhood structure

- A fighting structure F on I is a subset of N . The set of fighting structures is denoted by \mathfrak{F} .

According to this definition, there are n countries. Every particular pair $i-j$ (or $j-i$, which means the same) of countries i and j , $i \neq j$ belongs to $I^{(2)}$. There may (but need not) exist a common border between i and j . If i and j are indeed neighbors, $i-j$ belongs to the neighborhood structure N . In the absence of airplanes, fighting can occur only between neighbors. Therefore, a fighting structure F has to be a subset of the neighborhood structure N . This means that every fighting pair $i-j$ is made up of countries i and j that are neighbors.²⁷

A four-country case is depicted in Figure 2. It is chosen for the similarity with the chess board in Figure 1. Formally, we have the neighborhood structure $N = \{1-2, 2-3, 3-4, 4-1\}$ where countrys 1 and 3 are not neighbors and 2 and 4 are not neighbors.

4.2 Friends and enemies

We assume that countries can evaluate fighting structures by foreseeing the likely outcomes. We capture this idea by presupposing, for every country i , a payoff function

$$p_i : \mathfrak{F} \rightarrow \mathbb{R}$$

This means that country i attaches a real number to each fighting structure F from \mathfrak{F} . A higher number indicates that the fighting structure is better for country i . Here, payoff is just a word that stands for the advantages and

²⁷(Maoz et al.; 2007, pp. 104-105) define enmity networks and alliance networks which allow elegant computations of concepts like “enemy of enemy”, “friend of enemy”, or “enemy of friend”. They do not restrict attention to local warfare. See our remark to local warfare in the introduction.

disadvantages for the king of being engaged in a fighting structure. It may be measured in terms of land won or lost (more on this matter later).

We can now define the *Kautilyan* concepts of friendship (*mitra* means “friend” or “ally”) and enmity (*ari* is “enemy”). Of course, *Kautilya*’s understanding of friendship and enmity is a fluctuating one (see our remark in the conclusion). A friend has two characteristics. First, a friend is somebody against whom you do not fight. Second, you like to help a friend:

Definition 2. *Let I be a set of countries with fighting structure F . For three countries i , j , and k assume*

- (a) $i - j \notin F$,
- (b) $j - k \in F$,
- (c) $i - k \in N \setminus F$.

Country i is called a friend of j against k at F if

$$p_i(F \cup \{i - k\}) > p_i(F)$$

holds.

In the above definition, the meaning of (a) through (c) is given by

- (a) i and j do not fight each other,
- (b) j is engaged in a fight against k ,
- (c) i and k are neighbors, but do not actually fight against each other.

Then, country i is called a friend of j against k at F if i likes to start fighting against k (thus helping his friend i against his friend’s enemy k).

Similarly, one can be an enemy in two ways. Either you actually fight him or you like to do so:

Definition 3. *Assume countries i, j with $i - j \in N$. Country i is called an enemy of country j if one of two conditions hold:*

- *either i fights against j ($i - j \in F$),*
- *or, if i does not fight j , she would like to do so:*

$$i - j \notin F \Rightarrow p_i(F \cup \{i - j\}) > p_i(F).$$

Taking up the fighting structure depicted in Figure 2, country 1 is never a friend of country 2. This is due to the fact that 1 cannot attack 3. Also, country 1 is never an enemy of country 3. We think that these claims are in line with *Kautilyan* thought.

4.3 Strategies and equilibria

We now proceed to examine *Kautilya's* conjecture in a game-theoretic manner.²⁸ Mirroring the *Kautilyan* actions “remaining stationary” and “marching into battle”, we let every country decide which of its neighbors it wants to attack. We call the tuple of attack decisions a strategy. Second, we determine the fighting structure that results from these attack decisions. It is called an “induced fighting structure”. We find it natural to assume that two countries fight against each other if one of them attacks the other or both attack each other.

Definition 4. *Let I be a set of countries. Country i 's strategy s_i is a tuple with $|N(i)|$ entries where, for each $i - j \in N(i)$ (i.e., each of i 's neighbors) we have either $i \rightarrow j$ (i attacks j) or $i \nrightarrow j$ (i does not attack j). Let $s = (s_1, \dots, s_n)$ be a tuple of strategies (also known as strategy combination), one strategy for each country. Then, the induced fighting structure $F(s)$ contains $i - j$ if $i \rightarrow j$ or $j \rightarrow i$ hold.*

By s_{-i} we denote the strategy combination for all countries except country i , i.e., $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$. Let S denote the set of strategy combinations.

For example, in Figure 2, country 1 has four strategies:

- country 1 attacks both neighboring countries: $(1 \rightarrow 2, 1 \rightarrow 4)$
- country 1 attacks country 2, but not country 4: $(1 \rightarrow 2, 1 \nrightarrow 4)$
- country 1 attacks does not attack country 2, but attacks country 4: $(1 \nrightarrow 2, 1 \rightarrow 4)$
- country 1 does not attack any country: $(1 \nrightarrow 2, 1 \nrightarrow 4)$

Strategy combinations and the resulting fighting structures can be depicted in an intuitive manner. Figure 3 stands for the strategy combination $s = (s_1, \dots, s_4)$ with

$$\begin{aligned} s_1 &= (1 \rightarrow 2, 1 \rightarrow 4), \\ s_2 &= (2 \nrightarrow 1, 2 \rightarrow 3), \\ s_3 &= (3 \nrightarrow 2, 3 \nrightarrow 4), \\ s_4 &= (4 \rightarrow 1, 4 \nrightarrow 3) \end{aligned}$$

²⁸The interested reader can consult any one of the many textbooks on game theory, for example parts 1 and 2 in (Gibbons; 1992) or chapter 3 in (Dixit and Skeath; 1999).

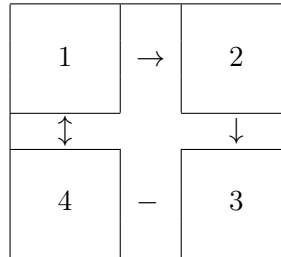


Figure 3: An induced fighting structure

and for the induced fighting structure

$$F(s) = \{1 - 2, 2 - 3, 4 - 1\}$$

The following definition prepares the ground for the application of game theory. So far, we have payoff functions $p_i : \mathfrak{F} \rightarrow \mathbb{R}$ that have the set of fighting structures as the domain. We need utility functions that have the set of strategy combinations as their domain. These utility functions are obtained via the induced fighting structures:

Definition 5. We define a utility function $u_i : S \rightarrow \mathbb{R}$ on S by $u_i(s) = p_i(F(s))$.

Thus, a utility function u_i uses strategy combinations as input. Here is the standard definition of a game-theoretic equilibrium (often also called Nash equilibrium):

Definition 6. A strategy combination s^* obeying

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

for all countrys $i \in I$ and all strategies $s_i \in S_i$ is called an equilibrium.

Thus, an equilibrium s^* is defined by the following property: No country $i \in I$ gains by not choosing s_i^* if the other countries choose s_{-i}^* . Differently put, unilateral deviation does not pay.

Definition 7. The strategy combination given by $i \rightarrow j$ for all $i, j \in I$ with $i - j \in N$ is called the trivial equilibrium.

Note that the trivial equilibrium is an equilibrium. Indeed, if only one country deviates, the fighting structure is not changed. Therefore, no country's payoff changes if that country alone stops attacking one or several of its neighbors.

4.4 Payoffs

In order to find additional equilibria (apart from the trivial one), we need to define specific payoff functions

$$p_i : \mathfrak{F} \rightarrow \mathbb{R}, i \in I$$

We assume that the payoffs obey the following assumptions:

- The size of each country is 1. This is the basic payoff and fighting power.
- If a country wins against another country, the winning country takes over the territory of the losing one.
- If several winning countries are involved, the losing country's territory is split evenly between the winning ones. This assumption is in line with *Kautilya's* recommendation to “urge a neighboring ruler to march into battle after concluding a pact, saying, ‘You should march in this direction, and I will march in that direction. The spoils shall be equal’ ”(Olivelle; 2013, p. 292; 7.6.2-3). (Note that *Kautilya* refers to a different neighborhood structure where three countries are adjacent to each other.)
- If a country is involved in two or more fights, its fighting power is split evenly.
- The fighting powers of several attackers are added. The relative fighting power determines the winner. If the fighting power is the same, the outcome is a “draw”.
- Fighting is costly. For each fight, the fighting power is reduced by $\delta > 0$. We assume that the cost of fighting is relatively small in comparison to a country's basic payoff and fighting power of 1 and let $\delta < \frac{1}{2}$.
- Basically, payoff equals fighting power. But each country prefers weaker neighbors to stronger ones. $\varepsilon > 0$ stands for the advantage of being stronger than neighbors while $-\varepsilon < 0$ represents the disadvantage of being weaker.

Focusing on our four-country example, these payoff assumptions can be translated into the following payoff function for country 1 (with analogous

payoffs for the other countries):

$$p_1(F) = \begin{cases} 1, & F = \emptyset & \text{(i)} \\ 1 + \varepsilon, & F = \{2 - 3\} \text{ or } F = \{3 - 4\} & \text{(ii)} \\ 1 - \varepsilon, & F = \{2 - 3, 3 - 4\} & \text{(iii)} \\ 1 - \delta, & F = \{1 - 2, 3 - 4\} \text{ or } F = \{1 - 4, 2 - 3\} \text{ or } F = N & \text{(iv)} \\ 1 - \delta - \varepsilon, & F = \{1 - 2\} \text{ or } F = \{1 - 4\} & \text{(v)} \\ \frac{3}{2} - \delta + \varepsilon, & F = \{1 - 2, 2 - 3\} \text{ or } F = \{1 - 4, 3 - 4\} & \text{(vi)} \\ 0, & F = \{1 - 2, 1 - 4\} & \text{(vii)} \\ 0, & F = \{1 - 2, 1 - 4, 2 - 3\} \text{ or } F = \{1 - 2, 1 - 4, 3 - 4\} & \text{(viii)} \\ 2 - \delta, & F = \{1 - 2, 2 - 3, 3 - 4\} \text{ or } F = \{1 - 4, 2 - 3, 3 - 4\} & \text{(ix)} \end{cases}$$

where (i) through (ix) can be seen from

- (i) no fighting
- (ii) two other countries fight
- (iii) 3 loses against 2 and 4
- (iv) all neighbors fight
- (v) 1 fights against 2 or 4
- (vi) 1 joins 3 to win against 2 or 4
- (vii) 1 loses against 2 and 4
- (viii) 1 and 2 lose, or 1 and 4 lose
- (ix) 1 and 4 win, or 1 and 2 win

For example, consider the second line where two other countries (2 and 3, or 3 and 4) fight. There is a draw between them and they suffer the costs of fighting δ . Hence, country 1 is stronger than the fighting pair of countries and his future fighting power is $1 + \varepsilon$. In line (vi), country 1 joins country 3 to attack country 2 (or country 4). 1 and 3 then share country 2's (or country 4's) territory. In the last line, both countries 2 and 3 (or both countries 3 and 4) lose so that country 1 obtains one extra full territory.

4.5 Identifying friends and enemies

We now present two theorems for the four-country case. First, we identify friends and enemies for this particular neighborhood structure. Second, we list all the equilibria. The proofs of the theorems are found in the appendix.

Theorem 1. *Assume the symmetric four-country structure together with the payoff p_1 given above and with $\varepsilon > \delta < \frac{1}{2}$. We find:*

- A** Country 1 is a friend of country 3 against 2 (against 4) at $\{2 - 3\}$ (at $\{3 - 4\}$) by $\delta < \frac{1}{2}$,²⁹
- B** Country 1 is an enemy of country 2 (of country 4) at $\{2 - 3\}$ (at $\{3 - 4\}$) by $\delta < \frac{1}{2}$,
- C** Country 1 is a friend of country 3 against 2 (against 4) at $\{2 - 3, 3 - 4\}$ by $\delta < 1 + \varepsilon$ (1 turns against either 2, or 4),
- D** Country 1 is an enemy of 2 (of 4) at $\{2 - 3, 3 - 4\}$ by $\delta < 1 + \varepsilon$ (1 turns against either 2, or 4),
- E** Country 1 is not a friend of country 3 against 2 at $\{1 - 4, 2 - 3, 3 - 4\}$ (or against 4 at $\{1 - 2, 2 - 3, 3 - 4\}$),
- F** Country 1 is not an enemy of country 2 (or country 4) at $F = \emptyset$.

Is the enemy of my enemy my friend? Consider the fighting structure $\{2 - 3\}$. Then, 1 is a friend of 3 against 2 (see A) and an enemy of 2 (see B). In that situation, the enemy of country 1's enemy is a friend. We can argue in a similar manner for the fighting structure $\{3 - 4\}$. However, E shows an example where *Kautilya's* conjecture does not hold:

- 4 is 1's enemy by $1 - 4$,
- 3 is 4's enemy by $3 - 4$, but
- 3 is not 1's friend because 1 is not prepared to attack 2.

Turn now to C versus D at fighting structure $F = \{2 - 3, 3 - 4\}$. If country 1 is a friend of country 3 against 2 at F , country 1 is an enemy of country 2. Again, the enemy of country 1's enemy is her friend.

Let us now report the equilibria for the symmetric four-country example, some of which are asymmetric:

Theorem 2. *In the symmetric four-country structure together with the payoff function p_1 and similarly for p_2 through p_4 and with $\varepsilon > \delta < \frac{1}{2}$, we have 20 equilibria:*

- a)** *the trivial equilibrium s^* leading to $F(s^*) = N$,*
- b)** *the no-attack equilibrium s^* given by $F(s^*) = \emptyset$,*

²⁹For $\delta > \frac{1}{2}$, country 1 would not like to attack country 3 at $\{2 - 3\}$ and hence, country 1 would not be a friend of country 3 against 2 at $\{2 - 3\}$.

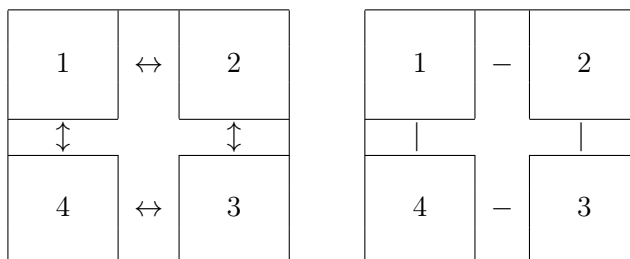


Figure 4: Allout fighting and peace

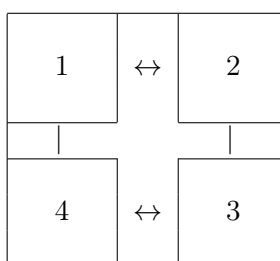


Figure 5: Two fighting pairs

- c) *the equilibrium s^* with fighting pairs 1–2 and 3–4 resulting from mutual attacks given by*

$$\begin{aligned} s_1^* &= (1 \rightarrow 2, 1 \nrightarrow 4), \\ s_2^* &= (2 \rightarrow 1, 2 \nrightarrow 3), \\ s_3^* &= (3 \nrightarrow 2, 3 \rightarrow 4), \\ s_4^* &= (4 \nrightarrow 1, 4 \rightarrow 3) \end{aligned}$$

together with the analogous equilibrium s^ with two fighting pairs 1–4 and 2–3,*

- d) *the $2 \times 2 = 4$ asymmetric equilibria s^* given by $F(s^*) = \{1-2, 2-3, 3-4\}$ and $2 \rightarrow 3$ and $3 \rightarrow 2$, where 1–2 (and also 3–4) may come about by both countries attacking or country 1 attacking country 2 (country 4 attacking country 3), together with the analogous $3 \times 4 = 12$ equilibria with no fighting between 1 and 2, 2 and 3, or 3 and 4, respectively.*

A few comments on these equilibria are in order. a) We have mentioned the trivial equilibrium in section 4.3. It is depicted in the left-hand side of Figure 4. b) The no-attack equilibrium or “peace” is a strategy combination where no country attacks any other (see the right-hand side of Figure 4). If

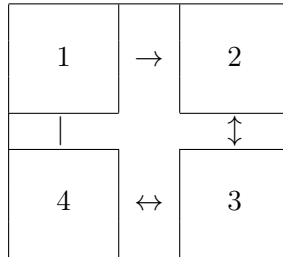


Figure 6: The horseshoe equilibrium

one country alone attacks any one country, the attacking country is worse off because his payoff is $1 - \delta - \varepsilon < 1$ (compare (v) and (i) in p_1 above). If one country attacks both its neighbors, its payoff is reduced to zero (see (vii) in p_1). c) Similar to the trivial equilibrium, we can have mutual attacks by disjoint pairs of countries (see Figure 5). Payoffs are $1 - \delta$ for all countries (see (iv) in p_1). If 1 attacks 2 and 2 attacks 1, neither of them can stop fighting by a unilateral action. If, in that situation, 1 attacks not only country 2, but also its other neighbor 4, the fighting structure $F = \{1 - 2, 1 - 4, 3 - 4\}$ results, again with zero payoff for country 1 (see (viii) in p_1). d) It is not difficult to see that fighting structures like $\{1 - 2, 2 - 3, 3 - 4\}$ can also be upheld in equilibrium. This horseshoe equilibrium is depicted in Figure 6. Here, countries 2 and 3 fight attack each other and are attacked from their other respective neighbors. Country 1, together with country 4, manages to overwhelm countries 2 and 3 (see (ix) in p_1). It does not matter whether the overwhelmed countries also attack. In the figure, country 3 attacks country 4, while country 2 does not attack country 1.

As in E of theorem 1, equilibria such as d) are in some contrast to *Kautilya's* conjecture. On the one hand, country 1 attacks his direct neighbor 2 (i.e., 1 is a friend of country 3 against 2). While seemingly in line with *Kautilya's* conjecture, matters are really more complicated: by not attacking country 4, country 1 indirectly (with the help of 4) also attacks country 3. On the other hand, country 1 does not attack his direct neighbor 4 (i.e., 1 is not a friend of country 3 against 4). Note that his last observation does not openly contradict *Kautilya's* conjecture because 1 is not 4's enemy.

5 Conclusions

The chess historian (Syed; 1995, pp. 69-70) summarizes her point of view in the following manner (the numbering is added by the current author):

1. The oldest *caturanga* was of the two-king variety.
2. It served the didactic purpose of practicing war strategies.
3. Dice were not used, intelligence alone determined the outcome.
4. In due time, onlookers wished to join and hence four-king chess was invented.
5. This chess for four players was a popularized variant, did not (necessarily) belong to the court anymore and did not (necessarily) serve the didactic purpose of teaching strategy in peace and war.

(Syed; 1995, p. 70) comments: “When [four-king chess] was not about teaching the art of war any more, the cognitive ability could be combined with the contingency of dice.”

While we do not wish to argue in favor or against the development sketched by Syed, we do raise objections against her comment. In our mind, the use of dice stands for an important aspect of managing war and peace. Repeating *Kautilya*’s (and Olivelle’s words): the divine realm, good and bad fortune, the invisible, what is beyond the range of thought, uncertainty. With respect to dice, we sympathize with (Schädler; 1999, p. 145) who criticizes the attitude pioneered by van der Linde according to which chess as a pure game of strategy is considered as intellectually and morally superior above chess with dice.

It is important to remember that the dice outcomes could be related to spellpower or to dexterity (see, for example, (Lüders; 1907, pp. 4–9, 29, 57–60)) rather than to mere luck. One might conjecture that a king or any other army ruler might try to get the particular parts of the army in motion, but that he may fail sometimes. The dice (that regulate whose pieces to move) might stand for this problem? Jacob Schmidt-Madsen (private communication) criticizes this viewpoint: “That dice can be considered a representation of daiva is certainly a well-attested fact — as seen, for example, in the religious game of snakes and ladders³⁰ — but their apparent use in four-handed chess has always struck me as a little odd. Rolling a die to determine which piece is allowed to move smacks more of gambling than of daiva to me (if I may be allowed to distinguish between the two). I mean, which army general would not have control over the movement of his troops? Would not the daivic interpretation be more plausible if it was applied to the outcome of

³⁰Snakes and Ladders is a children’s game with dice that is “derived from a traditional Indian game of some antiquity” according to (Topsfield; 1985, p. 203).

conflicts between pieces, or at least to the span of their movement as in most other dice-controlled board games?” These arguments are surely convincing. However, the rules of games (as they develop over time) have finally to be fixed in a manner as to make playing satisfactory. Since we do not understand the rules of four-king chess sufficiently well, we cannot exclude that the specific use of dice just served that purpose better than other uses.

Leaving the issue of dice, we mention the possibility (argued for by (Syed; 2001, pp. 10–11, 52)) that a didactic model, a sort of “Kriegsübung im Sandkasten”, which also carried the name *caturanga*, formed a connecting link between real-world warfare and two-king chess. (Schlingloff; 2009, p. 677) contradicts. One of his arguments is that the movements of chess pieces do not relate to the tactical manoeuvres on the battlefield. That is surely true. But the more general ideas of how to position troops or of how to deal with enemies and allies in situations of more than two kings may still hold, on a chess board as on a battlefield.

Jacob Schmidt-Madsen (private communication) notes that the game was not, perhaps, developed as a reflection of *Kautilya*’s theory. More generally, (Holländer; 1994, pp. 19–24) postulates that chess does not (necessarily) reflect cultural or social circumstances. Relatedly, (Burckhardt; 1969) points out that chess may symbolize heavenly bodies, four seasons, the dualism inherent in *yin* and *yan* etc. In the same vein, (Schädler; 2005, pp. 256–264) argues that board games in ancient civilizations are typically otherworldly oriented and that an Indian war game *caturanga* would be an outlier (calling out for an explanation). If, then, chess was not meant to reflect warfare, why would the pieces have an apparent link with Indian troops? (Schädler; 2005, pp. 264–265) advances the interesting thesis that this was done for mnemotechnical reasons. The different kinds of moves could easily be remembered by linking them to specific troops. There is, in our mind, no way to come to a definite answer. Whether or not otherworldly motives were crucial initially to bring Indian chess about is not really important for this paper and for the modest thesis defended here. Again, once a game is played the rules develop not only in line with a symbolic meaning (that may change over time) but certainly also in a manner to make playing satisfactory.

A central concern of this paper is Thieme’s dictum that four-king chess is “von [...] wirklichkeitsfremder Konstruktion”. We support Ghosh’s thesis that Indian four-king chess reflects *Kautilya*’s *maṇḍala* model. Indeed, four-king chess is as “unrealistic” as formal models in economics and elsewhere tend to be. It was deemed to be realistic enough so that princes could be taught the tricks of coping with friendly and unfriendly countries. In a similar manner, (Wiese; 2012) claims that Indian princes were meant to

learn backward induction by way of animal tales.

According to our game-theoretic analysis, terms like friend or enemy are defined only in terms of payoff (land to be occupied, power, ...). This way of thinking, however, is not foreign to Hindu political thought. This is clear from nearly every page of the *Arthaśāstra*. (Zimmer; 1969, p. 89) observes that Indian political thought was characterized by “cold-blooded cynical realism and sophistication”. He also finds that “ancient Hindu political wisdom” brings about “the cold precision of a kind of political algebra, certain fundamental natural laws that govern political life, no matter where” (p. 90). To us, Zimmer’s remarks are fitting for both the actions recommended within the *maṇḍala* model and for the use of game theory. Similarly, in a comment on *mitrasimhāsana* (see section 2.3), (Weber; 1873, p. 72) observes “an unfaithful politics of Indian chieftains” (German: “treulose Politik indischer Fürsten”).

The formal parts of our paper provide the instruments to discuss *Kauṭilya*’s *maṇḍala* theory. Loosely speaking, *Kauṭilya*’s conjecture holds most of the time. This is interesting by itself and also relevant to four-king chess. Apparently, the ally is found across the diagonal, in line with *Kauṭilya*’s conjecture.

Our simple formal model necessarily falls short of *Kauṭilya*’s *maṇḍala* theory in many respects, some of which we like to point out:

1. *Kauṭilya*’s *maṇḍala* theory is clearly dynamic in nature. We try to capture the gist of his theory by way of a static model, where ε reflects the dynamic aspects. Still, a formal dynamic model would do more justice to *Kauṭilya* and might bring out results that our static version suppresses. In particular, a dynamic model will reveal how friendship may turn into enmity.
2. While we define a general framework within which *Kauṭilya*’s *maṇḍala* theory can fruitfully be analyzed and discussed (or so we flatter ourselves), we surely do not cover all important aspects even in a static model. For example, middle and neutral kings play an important role in *Kauṭilya*’s thinking. Future research may take up these complicating factors as well as neighborhood structures with uneven fighting power.

6 Appendix

A Proof of theorem 1

In order to show how the proofs work, we present a few examples.

A Applying Definition 2, country 1 is a friend of country 3 against country 2 at $\{2 - 3\}$ because

$$\begin{aligned} 1 - 3 &\notin F \text{ (we even have } 1 - 3 \notin N) \\ 2 - 3 &\in F \text{ (we even have } F = \{2 - 3\}) \\ 1 - 2 &\in N, 1 - 2 \notin F \end{aligned}$$

and

$$p_1(\{2 - 3\} \cup \{1 - 2\}) = \frac{3}{2} - \delta + \varepsilon > 1 + \varepsilon = p_1(\{2 - 3\})$$

holds. The inequality follows from $\delta < \frac{1}{2}$.

D Applying Definition 3, country 1 is an enemy of 2 at $\{2 - 3, 3 - 4\}$ because 1 does not fight 2, but would gain from doing so:

$$p_1(\{2 - 3, 3 - 4\} \cup \{1 - 2\}) = 2 - \delta > 1 - \varepsilon = p_1(\{2 - 3, 3 - 4\}).$$

E Applying Definition 2, 1 is not a friend of country 3 against 2 at $\{1 - 4, 2 - 3, 3 - 4\}$. While the conditions

$$\begin{aligned} 1 - 3 &\notin F \text{ (we even have } 1 - 3 \notin N) \\ 2 - 3 &\in F \\ 1 - 2 &\in N, 1 - 2 \notin F \end{aligned}$$

are met, country 1 prefers not to join the fighting by

$$p_1(N) = 1 - \delta < 2 - \delta = p_1(\{1 - 4, 2 - 3, 3 - 4\}).$$

The other claims can be shown in a similar fashion.

B Proof of theorem 2

- a) We show in the main text (after Definition 7) that the trivial equilibrium (where every country attacks each of its neighbors) is an equilibrium.
- b) If no country attacks any other country, the payoff for each country is 1. Unilaterally attacking one country leads to the smaller payoff of $1 - \delta - \varepsilon$. Attacking two countries is even worse, resulting in payoff 0.
- c) Under the c) equilibria, each player has the payoff $1 - \delta - \varepsilon$. Deviating does not pay. Consider country 1.
- If 1 stops attacking country 2, that does not change the resulting fighting structure.
 - If 1 decides to attack both 2 and 4, country 1 (together with country 4) gets a payoff of 0.
 - If 1 decides to not attack 2, but to attack 4 instead, its payoff is the same as under the previous bullet item.
- d) The set-up here is the fighting structure $\{1 - 2, 2 - 3, 3 - 4\}$ that results from countries 2 and 3 attacking each other. It is not relevant whether $1 - 2$ or $3 - 4$ result from unilateral or from mutual attacks. In this situation, the payoffs for the winners 1 and 4 are $2 - \delta$ and the payoff for the losers are 0. Observe:
- Due to the mutual attack between countries 2 and 3, their payoff does not change if one of them decides not to attack.
 - If country 2, does not attack country 1, country 1 might attack 4 rather than 2. In that case, country 1's payoff stays at $2 - \delta$. If country 2 attacks country 1, country 1 can only change the fighting structure by attacking country 4. However, that would lead to the fighting structure N and reduce country 1's payoff from $2 - \delta$ to $1 - \delta$.

Since the other countries situations are symmetric, the d) equilibria are confirmed.

Note, finally, that no further equilibria exist.

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